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P H I L O S O P H I C A L
T R A N S A C T I O N S.

XVII. *Theorems for computing Logarithms.* By the Rev.
John Hellins; *communicated by the Rev. Nevil Maske-*
lyne, D. D. F. R. S. and Astronomer Royal.

Read March 16, 1780.

THE utility of logarithms is so well known, that much need not be said upon it. In our days he must be a slender mathematician who does not know that they are useful, not only in trigonometry, navigation, astronomy, the calculations of compound interest

and annuities, but also in the finding of fluents, and the summation of infinite series.

Some of the greatest mathematicians that this kingdom ever produced, as Sir ISAAC NEWTON, Dr. HALLEY, Mr. COTES, and Mr. SIMPSON, have thought it not beneath them to improve the construction of logarithms, which strongly argues the utility of those artificial numbers, and may suggest to us that the construction of them cannot be much further improved.

Now, although we should be very diffident in our expectations of improvement in any part of the mathematics after it has been handled by such great men, yet, if the method of computing be in general long and tedious, or if there still remain any particular difficulty, I believe, no good reason can be given why every attempt to abridge the one, or remove the other, should be discouraged. The easy method of computing the logarithms of large numbers given in page 49. of Mr. SIMPSON's pamphlet on Trigonometry is a proof that those gentlemen, who were of opinion that nothing better was to be hoped for, or expected, than what they published on the subject in the beginning of this century, were mistaken. And the following theorems, inferior to none as to convergency, and useful in deducing the logarithms of great fractions from those of small ones, or
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the logarithms of small numbers from those of great ones, may be considered as another proof of the mistake before mentioned. I have only to add here, that these theorems are new to me; and if they are so to the public, I humbly presume they will be acceptable.

THEOREM I.

The log. of $\frac{p+q}{p} = 2 \times \log. \text{ of } \frac{2p+2q}{2p+q} + \log. \text{ of } \frac{\overline{2p+q}^2}{2p+q^2-qq}.$

DEMONSTRATION.

$$\begin{aligned} \frac{\overline{2p+q}^2}{2p+q^2-qq} \times \frac{\overline{2p+2q}^2}{2p+q^2} &= \frac{1}{2p+q^2-qq} \times \frac{\overline{2p+2q}^2}{1} = \frac{4pp+8pq+4qq}{4pp+4pq} = \\ \frac{pp+2pq+qq}{p \times p+q} &= \frac{p+q}{p}: \text{consequently, } \log. \frac{\overline{2p+q}^2}{2p+q^2-qq} + 2 \log. \frac{2p+2q}{2p+q} \\ &= \log. \frac{p+q}{p}. \quad Q. E. D. \end{aligned}$$

COROLLARY.

If $q=1$, and we write n for p , the theorem becomes $\log. \frac{n+1}{n} = 2 \log. \frac{2n+2}{2n+1} + \log. \frac{\overline{2n+1}^2}{2n+1^2-1}$, which expression perhaps is of more frequent use than that above.

THEOREM II.

$$\text{Log. } \frac{p+q}{p} = 2 \log. \frac{2p+q}{2p} - \log. \frac{\overline{2p+q}^2}{2p+q^2-qq}.$$

D E M O N S T R A T I O N .

$\frac{2p+2q}{2p+q} \times \frac{2p+q}{2p} = \frac{2p+2q}{2p} = \frac{p+q}{p}$: therefore, $\log. \frac{p+q}{p} =$
 $\log. \frac{2p+2q}{2p+q} + \log. \frac{2p+q}{2p}$. But it has been proved above,
 that $\log. \frac{p+q}{p} = 2 \log. \frac{2p+2q}{2p+q} + \log. \frac{\overline{2p+q}^2}{2p+q^2-qq}$. If now we
 take this equation from twice the last there will remain
 $2 \log. \frac{p+q}{p} - \log. \frac{p+q}{p} = 2 \log. \frac{2p+2q}{2p+q} + 2 \log. \frac{2p+q}{2p} - 2 \log.$
 $\frac{2p+2q}{2p+q} - \log. \frac{\overline{2p+q}^2}{2p+q^2-qq}$: that is, $\log. \frac{p+q}{p} = 2 \log. \frac{2p+q}{2p} - \log.$
 $\frac{\overline{2p+q}^2}{2p+q^2-qq}$. Q. E. D.

C O R O L L A R Y .

Putting $q=1$, and $n=p$, as above, we have

$$\log. \frac{n+1}{n} = 2 \log. \frac{2n+1}{2n} - \log. \frac{\overline{2n+1}^2}{2n+1^2-1}.$$

I shall now set down some examples of the use of these theorems beginning with theorem 1.

The first example of the utility of this theorem may be in computing the logarithm of the number 2.

It is well known to mathematicians, that the computation of this logarithm was formerly a very laborious

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task :

task : and although the work be much shortened by help of the converging series invented by the illustrious Sir ISAAC NEWTON, still the logarithm of 2 has not been directly computed without many figures by any theorem I have yet seen. The easiest computation of it that has come to my hands is in page 44. of the late ingenious Mr. THOMAS SIMPSON's pamphlet on Trigonometry and logarithms. His series consists of the powers of $\frac{1}{3}$.

If now we put $n = 1$ in the theorem ($\log. \frac{n+1}{n} = 2 \log. \frac{2n+2}{2n+1} + \log. \frac{(2n+1)^2}{(2n+1)^2 - 1}$) we shall have $\log. \frac{2}{1} = 2 \log. \frac{4}{3} + \log. \frac{9}{8}$. Here then the fractions, whose odd powers are to be used, are $\frac{1}{3}$ and $\frac{1}{8}$; consequently, in the series formed from $\frac{1}{3}$, about one half of the number of terms taken by Mr. SIMPSON will give the result true to as many places of figures as his; and, from the fraction $\frac{1}{8}$, much fewer terms will suffice. To show how fast these series converge I will set down of each terms, enough to give the logarithm of 2 true to ten places of figures.

The odd powers of $\frac{2}{3}$ divided by their
respective indices.

1st,	0·14285714286
3d,	0·00097181730
5th,	0·00001189980
7th,	0·00000017347
9th,	0·00000000275
11th,	0·00000000004

The sum, 0·14384103622 is $\frac{1}{2}$ l. of $\frac{2}{3}$.

$$\begin{array}{r} 4 \\ \hline 0·5753641488 \text{ twice l. } \frac{2}{3}. \end{array}$$

The odd powers of $\frac{2}{3}$ divided by their
respective indices.

1st,	0·05882352941
3d,	0·00006784721
5th,	0·00000014086
7th,	0·00000000035

The sum, 0·05889151783 is $\frac{1}{2}$ l. of $\frac{2}{3}$.

$$\begin{array}{r} \text{Log. } \frac{2}{3}, \quad 0·11778303566 \\ 2 \text{ log. } \frac{2}{3}, \quad 0·57536414488 \\ \hline \text{Log. of 2. } 0·69314718054 \end{array}$$

But it is obvious, that this operation gives not only the logarithm of 2 but that of 3 also: for the logarithm of 4 being given from that of 2, and the logarithm of $\frac{4}{3}$ computed above, the logarithm of 3 is had, being = log. of 4 - log. of $\frac{4}{3}$.

$$\begin{array}{r} \text{Log. of 4} \quad 1·38629436108 \\ \text{Log. of } \frac{4}{3} \quad 0·28768207244 \\ \hline \text{Log. of 3} \quad 1·09861228864 \end{array}$$

Other examples of the use of these theorems in shewing how easily the logarithms of great fractions are derived from those of small ones.

If the logarithm of $\frac{64}{63}$ were given, or computed, we may very easily find the logarithm of $\frac{32}{31}$: for (by the-
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orem 1.) $2 \log. \frac{64}{63} + \log. \frac{6 \cdot 31^2}{6 \cdot 31^2 - 1} = \log. \frac{3^2}{3^1}$. Here the fraction, whose odd powers are to be used in the series, is $\frac{1}{7937}$, and the very first term of it, will give the logarithm true to twelve places of figures.

Again, if the logarithm of $\frac{16}{15}$ were to be computed from that of $\frac{3^2}{3^1}$ found above, we should have $2 \log. \frac{3^2}{3^1} + \log. \frac{31^2}{31^2 - 1} = \log. \frac{16}{15}$. Here the fraction to be used in the series is $\frac{1}{1921}$, the first term of which will give the logarithm true to ten places of figures.

In like manner, from the logarithm of $\frac{16}{15}$ we may find that of $\frac{8}{7}$; from logarithm of $\frac{8}{7}$ that of $\frac{4}{3}$; and from the logarithm of $\frac{4}{3}$ that of $\frac{2}{1}$, as is done above. The respective fractions for the series will be $\frac{1}{449}$, $\frac{1}{97}$, and $\frac{1}{17}$.

Thus far the fractions I have taken have even numbers for their numerators; let us now take one whose numerator is an odd number $\frac{9}{8}$. Here n being $= 3^{\frac{1}{2}}$, $\log. \frac{9}{7} (\frac{4^{\frac{1}{2}}}{3^{\frac{1}{2}}}) = 2 \log. \frac{9}{8} + \log. \frac{64}{63}$; and the fraction whose odd powers are to be used is $\frac{1}{127}$. Hence we have the log. of $\frac{8}{7}$ (for $\frac{9}{7} \div \frac{9}{8} = \frac{8}{7}$) and may proceed to find the logarithm of 2 as above. But the logarithm of $\frac{8}{7}$ may be directly derived from the equation thus: the equation in other terms is, $\log. 9 - \log. 7 = 2 \log. 9 - 2 \log. 8 + \log. \frac{64}{63}$; then, by transposition, $\log. 8 - \log. 7 = \log. 9 - \log. 8 + \log. \frac{64}{63}$; or $\log. \frac{8}{7} = \log. \frac{9}{8} + \log. \frac{64}{63}$.

But when the numerator of the fraction, whose logarithm is given, is odd, theorem 2. is more commodious. For taking $\frac{9}{8}$, as before, we have $2 \log. \frac{9}{8} - \log. \frac{81}{80} = \log. \frac{5}{4}$, where the fraction to be involved is $\frac{1}{161}$. Again, $2 \log. \frac{5}{4} - \log. \frac{25}{24} = \log. \frac{3}{2}$, where the fraction is $\frac{1}{49}$. And $2 \log. \frac{3}{2} - \log. \frac{9}{8} = \log. \frac{2}{1}$, where we have only to take the difference of logarithms, as the logarithm of $\frac{9}{8}$ as well as that of $\frac{3}{2}$ is given.

All the above calculations are of hyperbolic logarithms; but the same theorems hold good for Mr. BRIGGS'S, or any other. I will give an example in the computation of BRIGGS'S logarithm of 7 from others already known.

Let the logarithms of 100, 99, and 50, be given; then (by theorem 1.) $2 \log. \frac{100}{99} + \frac{99!^2}{99!^2 - 1} = \log. \frac{50}{49}$, or $\log. \frac{100}{99} + \frac{1}{2} \log. \frac{99!^2}{99!^2 - 1} = \frac{1}{2} \log. \frac{50}{49}$; and then $\frac{1}{2} \log. 50 - \frac{1}{2} \log. \frac{50}{49} = \frac{1}{2} \log. 49 = \log. 7$.

$$\text{Log. of } \frac{100}{99} \quad - \quad - \quad - \quad 0.00436480540245$$

$$\frac{1}{2} \log. \text{ of } \frac{99!^2}{99!^2 - 1} \left(= \frac{0.43429, \&c.}{19601} = \right) 0.00002215675128$$

$$\frac{1}{2} \log. \text{ of } \frac{50}{49} \quad - \quad - \quad - \quad 0.00438696215373$$

$$\frac{1}{2} \log. \text{ of } 50 \quad - \quad - \quad - \quad 0.84948500216801$$

$$\text{Log. of } 7 \quad - \quad - \quad - \quad 0.84509804001428$$

S C H O L I U M.

Neither the number 2, nor the fraction $\frac{64}{63}$ is chosen as the most advantageous to begin with in computing a table of logarithms; but they are taken as some of the first that occurred, to show the use of these theorems. Perhaps there are other instances in which they would be shown to much more advantage; but I hope their use will appear from the few examples given. They may indeed be transformed so as to be more commodious in particular cases, and there may be some others derived from them, one or two of which I will here put down.

It is evident from theorem 1. and 2. that $2 \log. \frac{2p+2q}{2p+q} + \log. \frac{2p+q^2}{2p+q^2-qq} = 2 \log. \frac{2p+q}{2p} - \log. \frac{2p+q^2}{2p+q^2-qq}$; consequently, $2 \log. \frac{2p+2q}{2p+q} + 2 \log. \frac{2p+q^2}{2p+q^2-qq} = 2 \log. \frac{2p+q}{2p}$, or, $\log. \frac{2p+q}{2p} = \log. \frac{2p+2q}{2p+q} + \log. \frac{2p+q^2}{2p+q^2-qq}$, which may be called theorem 3.

Again, this equation may be thus expressed: $\log. 2p+q - \log. 2p = \log. 2p+2q - \log. 2p+q + \log. \frac{2p+q^2}{2p+q^2-qq}$; and, by transposition, $2 \log. 2p+q = \log. 2p+2q + \log. 2p + \log. \frac{2p+q^2}{2p+q^2-qq}$, which may be called theorem 4. And this is, in effect, one of the theorems given by Dr. HAL-

LEY, in the Philosophical Transactions, N° 216, and of which the doctor said, it converges so very fast, that, in his opinion, nothing better was to be hoped.

There are yet some contrivances different from those mentioned in the beginning of SHERWIN's book of mathematical tables, or any other that have come to my hands, whereby the labour of computing a table of logarithms is shortened; but to explain them would require more time than my present situation affords me.

The observations and reasonings which led me to the discovery of the above theorems, I imagine, need not here be mentioned. Such as they are, I beg leave to lay them before the candid and skilful in these matters, in hopes that the invention will appear to them, as it does to me, a useful one.

It has, indeed, been objected, by a gentleman of my acquaintance, that improvements in the construction of logarithms cannot now be useful, because logarithms are already constructed.

I answer, that argument, if it has any weight, operates equally against Sir ISAAC NEWTON, Dr. HALLEY, Mr. COTES, Mr. SIMPSON, and several other ingenious mathematicians; for logarithms were invented, and tables of them constructed, before their time; so that if I should be thought to have misemployed my time in
improving

improving the computation of these artificial numbers, I have some consolation in thinking that I have therein followed the example of the very respectable company just mentioned.

I trust, however, that, with mathematicians, every improvement in calculation will be acceptable.

